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MUTUAL IRRADIATION SURFACES OF ELEMENTARY STRIPS OF COAXIAL CIRCULAR CONES

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Exact formulas for calculating the mutual irradiation surfaces and the coefficients of irradiation of the elementary strips of coaxial circular cones have been obtained. Formulas for the special case of calculating the elementary strips of coaxial circular cylinders are presented.

Keywords: *coaxial cones, elementary strips, mutual irradiation surfaces, irradiation coefficient.*

A radiating conical slot recuperator (RCSR) schematically represents two truncated coaxial circular cones. Inside the smaller cone, combustion products (gases) move, and, in the annular conical gap between the inner and outer cones, heated air circulates.

In the process of mathematical simulation of the heat-transfer processes in an RCSR, it is necessary to calculate the value of the mutual radiative-heat-exchange surface H_{12} between the inner-cone strip of width dx positioned at a height x and the outer-cone strip of width $d\xi$ positioned at a height ξ. The formula for calculating the quantity H_{12} was derived on the basis of the expression taken from [1] for the area of the mutual irradiation surface of the two surfaces F_1 and F_2 separated by a medium transparent for light beams:

$$H_{12} = \frac{1}{\pi}\int_{F_1} dF_1 \int_{F_2} \frac{\cos\beta_1 \cos\beta_2}{r^2}\, dF_2 \,, \tag{1}$$

where β_1 is the angle between the segment connecting the current points of the surfaces F_1 and F_2 and the normal to the surface F_1 at its current point, and β_2 is the analogous angle for the surface F_2. Here, F_1 and F_2 are elementary strips positioned on the inner conic surface at a height $x = |\mathrm{CD}|$ and on the outer surface at a height $\xi = |\mathrm{OD}|$ (Fig. 1). The radiation from the current point M of the inner cone strip falls on only a part of the outer-cone strip visible from the point M, namely, on the arc ATB with a current point T (Fig. 1b). Let us bring the origin of the cylindrical-coordinate system into coincidence with the point D and direct the z axis of the system along the line DK. The polar radii of the current points M, P, and N are equal, respectively, to (Fig. 1c):

$$\rho = R_1 - x\cot\alpha\,, \quad R = R_2 - \xi\cot\beta\,, \quad s = R_1 - \xi\cot\alpha\,. \tag{2}$$

The angular size of the arc ATB is $2\gamma = 2\arccos(s/R)$. The Cartesian coordinates of the point M are ρ, 0, x, and the polar coordinates of the point T are R, Θ, ξ (Fig. 1b). The Cartesian coordinates of the point T are $R\cos\Theta$ and $R\sin\Theta$, ξ; consequently, the vector $\mathbf{r} = \mathrm{MT} = \{R\cos\Theta - \rho;\ R\sin\Theta;\ \xi - x\}$. The modulus of this vector is equal to $\mathbf{r} = |\mathbf{r}| = \sqrt{x^2 + \xi^2 + R^2 + \rho^2 - 2x\xi - 2\rho R\cos\Theta}$. The unit vector of the normal to the surface of the inner cone at the point M is $\mathbf{n}_1 = \{\sin\alpha, 0, \cos\alpha\}$. Hence it follows that $\cos\beta_1 = [(R\cos\Theta - \rho)\sin\alpha + (\xi - x)\cos\alpha]/r$. The unit vector of the normal to the surface of the outer corner at the point T is $\mathbf{n}_2 = \{-\sin\beta\cos\Theta;\ -\sin\beta\sin\Theta;\ \cos\beta\}$. It follows that $\cos\beta_2 = [(R - \rho\cos\Theta)\sin\beta + (\xi - x)\cos\beta]/r$. The region of integration over F_2 is $\Theta \in [-\gamma;\ \gamma]$, $dF_2 = Rd\Theta d\xi/\sin\beta$. Since the integration element is independent of the polar angle of the point M, the integration over the region F_1 is equivalent to the multiplication of the inner integral on the right side of formula (1) into the measure of

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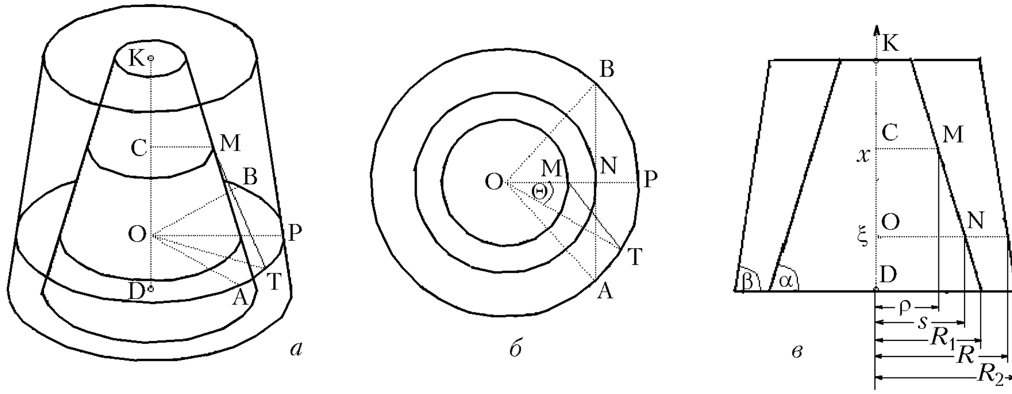


Fig. 1. For the derivation of the formula for calculating H_{12}: a) view of two conical surfaces in the axonometric projection; b) top view; c) view in the axial section.

the region F_1, i.e., by $2\pi\rho dx/\sin\alpha$. The integration function is even with respect to the variable Θ. From the aforesaid it follows that

$$H_{12} = 4\rho R dx d\xi \int_0^\gamma \frac{[R\cos\Theta - \rho + (\xi - x)\cot\alpha]\,[R - \rho\cos\Theta + (\xi - x)\cot\beta]}{(x^2 + \xi^2 + R^2 + \rho^2 - 2x\xi - 2\rho R\cos\Theta)^2}\, d\Theta\,. \tag{3}$$

Let us rearrange expression (3):

$$H_{12} = dx d\xi \int_0^\gamma \frac{(\cos\Theta + a)\,(b - \cos\Theta)}{(k - \cos\Theta)^2}\, d\Theta\,, \tag{4}$$

where

$$a = [(\xi - x)\cot\alpha - \rho]/R\,;\quad b = [(\xi - x)\cot\beta + R]/\rho\,;\quad k = \left[R^2 + \rho^2 + (\xi - x)^2\right]/(2R\rho)\,. \tag{5}$$

Let

$$H_{12} = h_{12} dx d\xi\,. \tag{6}$$

The quantity h_{12} on the right side of formula (4) represents an integral calculated using the universal trigonometrical substitution: $t = \tan(\Theta/2)$, $\Theta = 2\arctan(t)$, $\cos\Theta = (1 - t^2)/(1 + t^2)$, $d\Theta = 2dt/(1 + t^2)$. On elementary rearrangements, we obtain

$$h_{12} = 2q \int_0^{\tan(\gamma/2)} \frac{(t^4 + mt^2 + n)\, dt}{(t^2 + p)^2\,(t^2 + 1)}\,. \tag{7}$$

Here

$$m = (a + 1)/(a - 1) + (b - 1)/(b + 1)\,;\quad n = [(a + 1)\,(b - 1)/[(a - 1)\,(b + 1)]\,;$$

$$p = (k - 1)/(k + 1)\,;\quad q = (a - 1)\,(b + 1)/(k + 1)^2\,. \tag{8}$$

The integration function in formula (7) is a proper rational fraction; it can be represented as the sum of the simplest fractions

$$\frac{t^4 + mt^2 + n}{(t^2 + p)^2 (t^2 + 1)} = \frac{A_1 t + B_1}{(t^2 + p)^2} + \frac{A_2 t + B_2}{t^2 + p} + \frac{A_3 t + B_3}{t^2 + 1} . \tag{9}$$

Reducing the fractions on the right side of equality (9) to a common denominator and equating the coefficients of the equal exponents of the variable t in the numerators on the left and right sides of the equality, we obtain two systems of linear algebraic equations for determining the expansion coefficients A_1, A_2, A_3, B_1, B_2, and B_3:

$$A_1 + A_3 = 0 , \quad A_1 + (p + 1) A_2 + 2pA_3 = 0 , \quad A_1 + pA_2 + p^2 A_3 = 0 ;$$

$$B_1 + B_3 = 1 , \quad B_1 + (p + 1) B_2 + 2pB_3 = m , \quad B_1 + pB_2 + p^2 B_3 = n .$$

Solution of these systems gives

$$A_1 = A_2 = A_3 = 0 , \quad B_1 = (p^2 - mp + n)/(1 - p) ,$$
$$B_2 = (p^2 - 2p - n + m)/(1 - p)^2 , \quad B_3 = (n - m + 1)/(1 - p)^2 . \tag{10}$$

The integral on the right side of formula (7) is calculated using expansion (9) with account for the fact that $\tan (\gamma/2) = \sqrt{\frac{R - s}{R + s}}$. As a result, we eventually obtain

$$h_{12} = q \left\{ \frac{B_1 \sqrt{R^2 - s^2}}{p (R - s + p (R + s))} + \frac{B_1 + 2pB_2}{p\sqrt{p}} \arctan \sqrt{\frac{R - s}{p (R + s)}} + B_3 \arctan \sqrt{\frac{R^2}{s^2} - 1} \right\} . \tag{11}$$

Formulas (2), (5), (6), (8), (10), and (11) in total allow one to exactly calculate the values of the mutual irradiation surface of the elementary strips of coaxial circular cones. In this case, since the quantity h_{12} is dimensionless, it may be assumed that $R_1 = 1$ in formulas (2) and the quantities R_2, x, and ξ are determined in fractions of R_1. However, the values of the differentials dx and $d\xi$ in formula (6) should be determined in meters.

Using the relations for the mutual irradiation surface and the irradiation coefficients taken from [1], we obtain the formulas for calculating:

the coefficient of irradiation of an elementary strip of the outer cone by an elementary strip of the inner cone

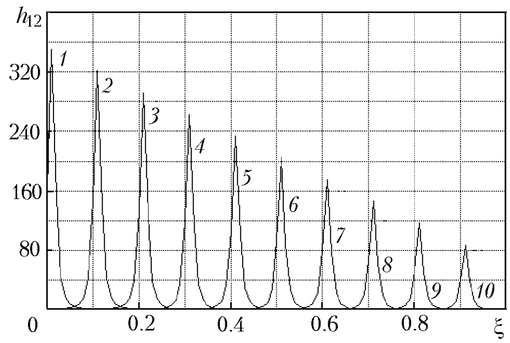
$$d\varphi_{d1-d2} = \frac{h_{12} \sin \alpha d\xi}{2\pi (R_1 - x \cot \alpha)} ; \tag{12}$$

the coefficient of irradiation of an elementary strip of the inner cone by an elementary strip of the outer cone

$$d\varphi_{d2-d1} = \frac{h_{12} \sin \beta dx}{2\pi (R_2 - \xi \cot \beta)} . \tag{13}$$

On the assumption that $\alpha = \beta = \pi/2$, the following formulas for coaxial circular cylinders are obtained:

$$h_{12} = \frac{2 (Y^2 - X^2 - 1)\sqrt{Y^2 - 1}}{[(Y + 1)^2 + X^2] (X^2 + Y^2 - 1)} - \arctan \sqrt{Y^2 - 1} + \frac{2}{\sqrt{(Y - 1)^2 + X^2} \, [(Y + 1)^2 + X^2]^{3/2}}$$
$$\times \left\{ (Y^2 - 1)^2 + 2 (Y + 1)^2 X^2 + X^4 + \frac{2Y [(Y^2 - 1)^2 - X^4]}{(Y - 1)^2 + X^2} \right\} \arctan \sqrt{\frac{(Y - 1) [(Y + 1)^2 + X^2]}{(Y + 1) [(Y - 1)^2 + X^2]}} . \tag{14}$$



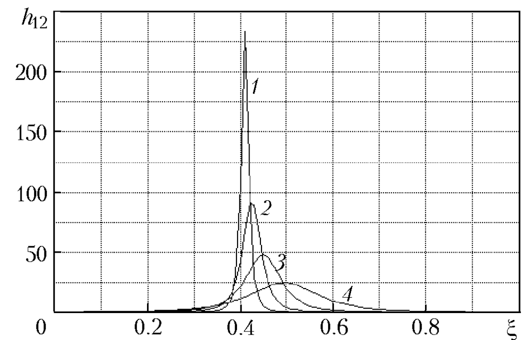


Fig. 2. Graphs of the functions $h_{12}(\xi)$ at $R_1 = 1$ m, $R_2 = 1.02$ m, and $\alpha = \beta = 50^{\circ}$ for different values of x: $x = 0$ (1), 0.1 (2), 0.2 (3), 0.3 (4), 0.4 (5), 0.5 (6), 0.6 (7), 0.7 (8), 0.8 (9), and 0.9 (10). ξ, m.

Fig. 3. Graphs of the functions $h_{12}(\xi)$ at $R_1 = 1$ m; $x = 0.4$; $\alpha = \beta = 50^{\circ}$ for different values of $R_2 = 1.02$ m (1), 1.05 (2), 1.1 (3), and 1.2 m (4). ξ, m.

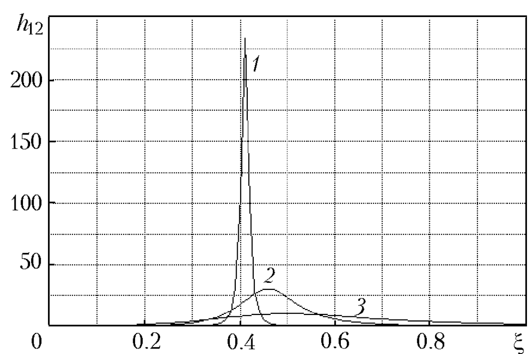


Fig. 4. Graphs of the functions $h_{12}(\xi)$ at $x = 0.4$, $R_1 = 1$ m, $R_2 = 1.02$ m, and $\alpha = 50^{\circ}$ for different values of the angle β: $\beta = 50^{\circ}$ (1), 60° (2), and 80° (3). ξ, m.

Here, $X = |x - \xi| / R_1$ and $Y = R_2 / R_1$.

Formulas (12) and (13) for the elementary strips of coaxial circular cylinders take the form

$$d\varphi_{d1-d2} = \frac{h_{12} d\xi}{2\pi R_1}, \quad d\varphi_{d2-d1} = \frac{h_{12} dx}{2\pi R_2},$$

where h_{12} is calculated by formula (14).

An analysis of the graphs presented in Fig. 2 showed that 99% of the radiation of a strip of the inner cone falls on the outer-cone strip of width $\sim 0.1 R_1$. The center of the strip, on which the main portion of the radiation falls, is shifted upwards by approximately $0.01 R_1$ relatively to the position of the inner-cone strip. The maximum value of $h_{12}(\xi)$ decreases with increase in the value of x since, at $\beta < 90^{\circ}$, an increase in x leads to a decrease in the value of the integration region on the right side of formula (7).

Figure 3 presents graphs of the function $h_{12}(\xi)$ for different values of R_2. When the value of R_2 increases, the width of the outer cone strip, on which 99% of the radiation of the inner-cone strip falls, increases: it comprises approximately $0.1 R_1$ at $R_2 = 1.02$ m, $0.15 R_1$ at $R_2 = 1.05$ m, $0.3 R_1$ at $R_2 = 1.1$ m, and $0.45 R_1$ at $R_2 = 1.2$ m. The center of the strip, on which 99% of the radiation falls, is shifted upwards by the value $\Delta = (R - x(\cot\beta - \cot\alpha) - 1) \sin\beta \cos\beta$ relative to the height of the inner-cone strip. The maximum value of $h_{12}(\xi)$ decreases with increase in the quantity R_2 since, in this case, the width of the outer-cone strip, on which the main portion of the radiation falls, increases.

Figure 4 presents graphs of the function $h_{12}(\xi)$ for different values of the angle β. When the angle of inclination of the generatrix of the outer cone β increases, the width of the outer-cone strip, on which the main portion of the radiation of the inner-cone strip falls, increases. The maximum value of $h_{12}(\xi)$ decreases with increase in the value of β since this leads to an increase in the width of the outer-cone strip, on which the main portion of the radiation of the inner-cone strip falls.

CONCLUSIONS

1. The width of the outer-cone strip, on which 99% of the radiation of an inner-cone elementary strip falls, increases with increase in both the relative width of the conical slot between the inner and outer cones and the difference between the angles of inclination of the generatrices of the outer and inner cones.

2. For conical recuperators having typical geometric parameters (the width of the slot is equal to 0.02 of the radius of the gas-channel base and the angles of inclination of the generatrices $\alpha = \beta = 50^{\circ}$) the width of the outer-cone strip, on which 99% of the radiation of an inner-cone elementary strip falls, comprises 0.1 of the radius of the gas-channel base. Therefore, simplified mathematical models of the radiation slots of recuperators can be constructed on the assumption that all radiation of an elementary strip of the inner cone falls on the opposite outer-cone strip of equal width.

NOTATION

a, b, k, m, n, p, q, X, Y, dimensionless quantities; dx, $d\xi$, differential of length, m; dF, differential of area, m^2; F, surface of radiative heat exchange; H, mutual surface of radiative heat exchange, m^2; h, dimensionless function; **n**, unit vector of the normal to the surface of radiative heat exchange; **r**, vector directed from the radiating point to the irradiated point; r, modulus of the vector **r**, m; R, s, polar radii, m; R_1, radius of the inner-cone base, m; R_2, radius of the outer-cone base, m; x, distance along the axis of the cone from its base to the elementary strip of the inner cone, n; α, β, angles between the generatrices of the inner and outer cones and their bases, rad; γ, Θ, angles, rad; Δ, shift, m; φ, radiation coefficient, dimensionless quantity; ξ, distance along the axis of the cone from its base to the elementary strip of the outer cone, m; ρ, polar radius, m. Subscripts: 1, 2, inner and outer cones.

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